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Research on an Improper Fractional Integral

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study an improper fractional integral. Moreover, our result is a generalization of traditional calculus result.**

Keywords: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, improper fractional integral.**

I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as physics, biology, mechanics, electrical engineering, viscoelasticity, control theory, modelling, economics, etc [1-15]. However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivative. Other useful definitions include Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie type of R-L fractional derivative to avoid non-zero fractional derivative of constant function [16-20].

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study the following improper fractional integral:

$$\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(sin_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}2}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-2)}\right],$$

where $0<\alpha\leq 1$. In fact, our result is a generalization of traditional calculus result.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([21])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([22])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D^{\alpha}_{x}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

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and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Definition 2.3 ([23])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([24])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([25])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{9}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{10}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{11}$$

Theorem 2.6 (integration by parts for fractional calculus) ([26])**:** *Assume that $0 < \alpha \leq 1$, a, b are real numbers, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions, then*

$$\left({}_aI_b^{\alpha}\right)\left[f_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^{\alpha}\right)[g_\alpha(x^\alpha)]\right] = \left[f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)\right]_{x=a}^{x=b} - \left({}_aI_b^{\alpha}\right)\left[g_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^{\alpha}\right)[f_\alpha(x^\alpha)]\right]. \tag{12}$$

Theorem 2.7 ([27])**:** *If $0 < \alpha \leq 1$, then the improper α-fractional integral*

$$\left({}_0I_{+\infty}^{\alpha}\right)\left[sin_\alpha(x^\alpha) \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right] = \frac{\pi}{2}. \tag{13}$$

III. MAIN RESULT

In this section, we solve a fractional integral involving fractional trigonometric function.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, *then the improper* α*-fractional integral*

$$\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-2)}\right]=\frac{\pi}{2}. \quad (14)$$

Proof $\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-2)}\right]$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\otimes_\alpha\left({}_0D_x^{\alpha}\right)\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right)\right]$$

$$=\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\otimes_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right)\right]_{x=0}^{x=+\infty}$$

$$-\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right)\otimes_\alpha\left({}_0D_x^{\alpha}\right)\left[\left(sin_\alpha(x^\alpha)\right)^{\otimes_\alpha 2}\right]\right]$$

(by integration by parts for fractional calculus)

$$=0-0+\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\otimes_\alpha 2\cdot sin_\alpha(x^\alpha)\otimes_\alpha cos_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\otimes_\alpha sin_\alpha(2x^\alpha)\right]$$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\otimes_\alpha sin_\alpha(2x^\alpha)\otimes_\alpha\left({}_0D_x^{\alpha}\right)\left[2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]\right]$$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left[sin_\alpha(t^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-1)}\right]$$

$$=\frac{\pi}{2}\ .$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we solve an improper fractional integral. In addition, our result is a generalization of ordinary calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in engineering mathematics and fractional differential equations.

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